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A new Riemann–Hilbert problem in a model of stimulated Raman Scattering

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Abstract

The Riemann–Hilbert problem proposed in [2] for the integrable stimulated Raman scattering (SRS) model was shown to be solvable under an additional condition: the boundary data have to be chosen in such a way that a corresponding spectral problem has no spectral singularities. In the general case, it can be shown that a spectral singularity occurs at $k = 0$. On the other hand, the initial boundary value (IBV) problem for the SRS equations is known to be well posed: using PDE techniques, this has been established in [3]. Therefore, it seems natural to try to find a new RH problem that is solvable in the presence of arbitrary spectral singularities. The formulation of such a RH problem is the main aim of the paper. Then the solution of the nonlinear initial boundary value problem for the SRS equations is expressed in terms of the solution of a linear problem which is the Riemann–Hilbert problem for a sectionally analytic matrix function.

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1. Introduction

There are many publications devoted to the stimulated Raman scattering (SRS). Papers [1–3] are on the subject of the present paper. The paper [1] deals with a problem of Raman soliton generation from laser inputs in SRS. It was shown that the SRS equations, solved as a boundary value problem on the semi-line, do induce the generation of solitons by pairs, and that, after the passage of the pulses, the solitons are static in the medium. In particular, this paper provides the derivation of the SRS equations when group velocity dispersion is taken into account. The case of zero group velocity dispersion was studied in [2] under some additional assumptions which lead to a model of transient SRS. More information about different models of the SRS and their physical meaning can be found in references of the above cited papers.

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The phenomenon of stimulated Raman scattering is described by three coupled PDEs. In the transient limit these equation are integrable. The relevant physical problem can be formulated as initial boundary value problem (IBV) on a finite domain with trivial initial function (identically equal zero). The general IBV problem can be written in the form

$$2\mathrm{i}q_t = \mu, \qquad \mu_x = 2\mathrm{i}\nu q, \qquad \nu_x = \mathrm{i}(\bar{q}\mu - q\bar{\mu}), \qquad x \in (0, L), \quad t \in (0, T), \tag{1}$$

with initial function

$$q(x, 0) = u(x), \qquad x \in [0, L], \tag{2}$$

and boundary condition

$$\mu(0, t) = w(t), \qquad \nu(0, t) = v(t), \qquad t \in [0, T]. \tag{3}$$

Assuming that the functions $q(x, t) \in \mathbb{C}$, $\mu(x, t) \in \mathbb{C}$, $\nu(x, t) \in \mathbb{R}$ satisfy the SRS equations (1) on the finite domain $x, t \in ((0, L) \times (0, T))$, it has been shown [2] that the solution of these equations can be obtained by solving a matrix Riemann–Hilbert (RH) problem formulated in the complex k-plane. This was achieved by implementing a new method introduced in [4]. The method consists of performing simultaneously the spectral analysis of the two parts forming the Lax pair. It was also shown in [4] that the long-distance behaviour of the system is described by the underlying self-similar solution connected with the third Painleve transcendent. The Riemann–Hilbert problem, proposed in [2], is solvable under the additional condition: boundary data $v(t)$ and $w(t)$ have to be chosen in such a way that corresponding spectral problem has spectral singularities nowhere. If $v(T) \neq -1$, it is easy to prove that the spectral singularity takes place at point $k = 0$. Besides, in the case of frequency mismatch between the pump and Stokes waves the eigenfunctions and spectral data have singularities that were also noted in [2]. On the other hand, the IBV problem for SRS equations is well posed that was established in [3] using PDE techniques.

Therefore, it is necessary to find a new RH problem which will be solvable without such a restriction, i.e. in the presence of arbitrary spectral singularities. We propose such a RH problem in section 5. A formulation of the suitable RH problem is the main aim of the paper. As a consequence of this formulation, the solution of nonlinear IBV problem (1)–(3) for SRS equations is expressed through the solution of a linear problem: the Riemann–Hilbert problem for sectionally analytic matrix functions (theorem 5.1).

Remark 1.1. The SRS equations admit a 'conservation low':

$$\frac{\partial}{\partial x}(\nu^2(x, t) + |\mu(x, t)|^2) = 0.$$

In what follows, we will put

$$\nu^2(x, t) + |\mu(x, t)|^2 \equiv 1.$$

2. Basic solutions of linear overdetermined equations

For studying the initial boundary value problem (1)–(3), we will use the simultaneous spectral analysis of the linear x-equation:

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x, t)\Phi,$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ -\bar{q}(x, t) & 0 \end{pmatrix} \tag{4}$$

and the linear t-equation

$$\Phi_t = \frac{\mathrm{i}}{4k}\widehat{Q}(x,t)\Phi, \qquad \widehat{Q}(x,t) = \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\bar{\mu}(x,t) & -\nu(x,t) \end{pmatrix}, \tag{5}$$

where $\Phi(x,t,k)$ is a 2×2 matrix-valued function and $k \in \mathbb{C}$ is a parameter. It is easy to verify that the overdetermined system of differential equations (4), (5) is compatible if and only if the functions $q(x,t), \mu(x,t), \nu(x,t)$ satisfy the SRS equations (1).

Let us rewrite equations (4) and (5) in the form

$$W_x = U(x,t,k)W, \tag{6}$$

$$W_t = V(x,t,k)W, \tag{7}$$

where $U(x,t,k)$ and $V(x,t,k)$ are matrices:

$$U(x,t,k) = Q(x,t) - \mathrm{i}k\sigma_3, \qquad V(x,t,k) = \frac{\mathrm{i}}{4k}\widehat{Q}(x,t),$$

given in terms of $q(x,t), \quad \mu(x,t), \quad \nu(x,t)$.

Lemma 2.1. *Let equations (7) and (8) be compatible for all $k \in \mathbb{C}$. Let $W(x,t,k)$ be a matrix satisfying the x-equation (7) for all t (the t-equation (8) for all x). Assume that $W(x_0,t,k)$ satisfies the t-equation (8) for some $x = x_0$ (the x-equation (7) for some $t = t_0$). Then $W(x,t,k)$ satisfies the t-equation (8) for all x (satisfies the x-equation (7) for all t).*

Proof. If $W = W(x,t,k)$ is a solution to (6), then $\hat{W}(x,t,k) = W_t - V(x,t,k)W$ is also the solution to (6). Indeed, $\hat{W}_x = U(x,t,k)\hat{W} + (U_t - V_x + [U,V])W = U(x,t,k)\hat{W}$. Since the matrices W and $\hat{W}$ are the solutions of the same equation (6), it follows that $\hat{W}(x,t,k) = W(x,t,k)C(t,k)$ for some $C(t,k)$ independent of x. By assumption, $\hat{W}(x_0,t,k) = 0$. Hence, $C(t,k) \equiv 0$ and thus $\hat{W}(x,t,k) \equiv 0$, which means that $W(x,t,k)$ satisfies the t-equation (7) for all x. The proof of the statement with x and t interchanged is similar. □

Let $q(x,t), \mu(x,t), \nu(x,t)$ be a solution of (1). Introduce $u(x) = \mu(x,0), w(t) = \mu(0,t)$ and $v(t) = \nu(0,t)$. Assume that $u(x) \in H^1[0,L], w(t) \in H^1[0,T]$ and $v(t) \in H^1[0,T]$. Also assume that $v^2(t) + |w(t)|^2 \equiv 1$. Then, equations (4) and (5) (equivalently, equations (6) and (7)) are compatible. In order to construct basic solutions (eigenfunctions) of (4), (5), let us define the matrix function $m_d(x,t)$ by

$$m_d(x,t) = m(x,t)d^{\sigma_3}(x,t), \tag{8}$$

where

$$m(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-\nu(x,t)} & \dfrac{\mathrm{i}\mu(x,t)}{\sqrt{1-\nu(x,t)}} \\ \dfrac{\mathrm{i}\bar{\mu}(x,t)}{\sqrt{1-\nu(x,t)}} & \sqrt{1-\nu(x,t)} \end{pmatrix}$$

and the diagonal matrix function $d^{\sigma_3}(x,t)$ is chosen in such a way that the matrix $m_d^{-1}(x,t)\dot{m}_d(x,t)$ to be off-diagonal, i.e., $\sigma_3\left(m_d^{-1}\dot{m}_d\sigma_3 + \sigma_3 m_d^{-1}\dot{m}_d = 0\right)$. Then, $Q(x,t)$ can be written in the form

$$Q(x,t) = m_d'(x,t)m_d^{-1}(x,t).$$

Here and in what follows, the dot and prime stand for the partial derivatives with respect to t and x, respectively. Note that $m(x,t)$ as well as $m_d(x,t)$ diagonalizes $\hat{Q}(x,t)$:

$$\hat{Q}(x,t) = -m(x,t)\sigma_3 m^{-1}(x,t) = -m_d(x,t)\sigma_3 m_d^{-1}(x,t).$$

By direct calculation,

$$d(x,t) = \exp\left\{\int_0^x \frac{\nu(\xi,t)\nu'(\xi,t) + \mu'(\xi,t)\bar{\mu}(\xi,t)}{2\nu(\xi,t)(1-\nu(\xi,t))}\,\mathrm{d}\xi + \int_0^t \frac{\nu(x,\tau)\dot{\nu}(x,\tau) + \dot{\mu}(x,\tau)\bar{\mu}(x,\tau)}{2(1-\nu(x,\tau))}\,\mathrm{d}\tau\right\}, \tag{9}$$

where d is normalized by $d(0,0) = 1$. Note that due to the identity

$$\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1,$$

the integrands in (9) are purely imaginary and thus $|d(x,t)| \equiv 1$.

Introduce $Z(x,t,k)$ by

$$\Phi(x,t,k) = m_d(x,t)Z(x,t,k),$$

where $\Phi(x,t,k)$ solves (4), (5). Then, $Z(x,t,k) = m_d^{-1}(x,t)\Phi(x,t,k)$ satisfies the equations

$$Z' + \mathrm{i}k m_d^{-1}(x,t)\sigma_3 m_d(x,t)Z = 0, \qquad \dot{Z} + \frac{\mathrm{i}\sigma_3}{4k}Z = -m_d^{-1}(x,t)\dot{m}_d(x,t)Z. \tag{10}$$

Equations (10) are gauge equivalent to (4), (5). The definition of the gauge equivalence can be found in [6]. The first equation in (10) has a form of the x-equation of the continuous model for the Heisenberg ferromagnet [6], whereas the second equation in (10) is the Dirac equation with the spectral parameter $\lambda = \frac{1}{4k}$.

The original equations (4), (5) and the gauge equivalent equations (10) admit the transformation operators [6, 5]: there are solutions $\Phi_0(x,t,k)$ and $\Phi_T(x,t,k)$ of equations (4), (5), which have the integral representations in the form

$$\Phi_0(x,t,k) = m_d(x,t)Z_0(x,t,k), \qquad \Phi_T(x,t,k) = m_d(x,t)Z_T(x,t,k),$$

where

$$Z_0(x,t,k) = \left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + k\int_{-x}^{x}\Gamma(x,y,t)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right)\left(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \int_{-t}^{t} M_0(t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right), \tag{11}$$

$$Z_T(x,t,k) = \left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + k\int_{-x}^{x}\Gamma(x,y,t)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right)\left(\mathrm{e}^{-\frac{\mathrm{i}\sigma_3}{4k}} + \int_{t}^{2T-t} M_T(t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right). \tag{12}$$

These solutions satisfy the conditions $\Phi_0(0,0,k) = m_d(0,0) = m(0,0)$ and $\Phi_T(0,T,k) = m_d(0,T)\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}$. The first factor in formulae (11) and (12) satisfies the x-equation (10) for all t, whereas the second factors satisfy the t-equation (10) for $x = 0$. By lemma (2), $Z_0(x,t,k)$ and $Z_T(x,t,k)$ are the solutions of both the equations in (10) for all x and t. In turn, $\Phi_0(x,t,k)$ and $\Phi_T(x,t,k)$ are the solutions of the original equations (4), (5). The existence of such representations, i.e., the existence of appropriate k-independent kernels $\Gamma(x,y,t)$, $M_0(t,s)$ and $M_T(t,s)$, can be proved following the scheme in [6] and [5].

The integral representations (11) and (12) imply the following properties of the matrices $\Phi_0(x,t,k)$ and $\Phi_T(x,t,k)$:

(1) $\Phi_0(x,t,k)$ and $\Phi_T(x,t,k)$ satisfy the x- and t-equations (4), (5);

(2) $\Phi(x,t,k) = \Lambda\bar{\Phi}(x,t,\bar{k})\Lambda^{-1}$, $k \in \mathbb{C}\backslash\{0\}$, where $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(3) $\det \Phi_0(x,t,k) = \det \Phi_T(x,t,k) \equiv 1, \quad k \in \mathbb{C}\setminus\{0\}$;
(4) the map$(x,t) \longmapsto \Phi(x,t,k)$ is absolutely continuous in x for fixed t and vice versa;
(5) the map $k \longmapsto \Phi(x,t,k)$ is analytic in $k \in \mathbb{C}\setminus\{0\}$;
(6) $\Phi_0(x,t,k)\,\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}} = m_d(x,t) + O(k) + O(k\,\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{2k}})$, $k \in \mathbb{C}\setminus\{0\}$; $\Phi_T(x,t,k)\,\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}} = m_d(x,t) + O(k) + O(k\,\mathrm{e}^{\frac{\mathrm{i}(t-T)\sigma_3}{2k}})$, $k \in \mathbb{C}\setminus\{0\}, k \to 0$

In items (2), (4) and (5), $\Phi(x,t,k)$ means $\Phi_0(x,t,k)$ or $\Phi_T(x,t,k)$. These eigenfunctions possess a 'good' (well-controlled) asymptotic behaviour as $k \to 0$; they will be used below for the construction of the matrix Riemann–Hilbert problem in the neighbourhood of $k = 0$.

Now we introduce another set of eigenfunctions of equations (4), (5), which possess a 'good' asymptotic behaviour as $k \to \infty$. The eigenfunction normalized by the condition $\Phi_1(0,T,k) = \mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}$ has the form

$$\Phi_1(x,t,k) = \left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{-x}^{x} K(x,y,t)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right)\left(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int_{t}^{2T-t} L_T(t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right), \tag{13}$$

where the first factor satisfies the x-equation (4) for all t, and the second factor satisfies the t-equation (5) for $x = 0$. By lemma 2, $\Phi_1(x,t,k)$ satisfies both equations (4) and (5). Since $\Phi_1(0,T,k) = \mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}$ and $\Phi_T(0,T,k) = m_d(0,T)\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}$, it follows that

$$\Phi_1(x,t,k) = \Phi_T(x,t,k)\,\mathrm{e}^{\frac{\mathrm{i}T\sigma_3}{4k}} m_d^{-1}(0,T)\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}.$$

Hence, $\Phi_1(x,t,k)$ possesses properties (1)–(5).

The integral representation (13) implies the following behaviour of $\Phi_1(x,t,k)$:

$$\Phi_1(x,t,k)\,\mathrm{e}^{\mathrm{i}kx\sigma_3} = I + O(k^{-1}) + O(k^{-1}\,\mathrm{e}^{2\mathrm{i}kx\sigma_3}), \qquad k \to \infty. \tag{14}$$

The eigenfunction $\Phi_2(x,t,k)$ normalized by the condition $\Phi_2(0,0,k) = I$ has the form

$$\Phi_2(x,t,k) = \left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{-x}^{x} K(x,y,t)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right)\left(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int_{-t}^{t} L_0(t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right). \tag{15}$$

It is related to $\Phi_0(x,t,k)$ by

$$\Phi_2(x,t,k) = \Phi_0(x,t,k)m^{-1}(0,0).$$

The eigenfunction $\Phi_2(x,t,k)$ satisfies properties (1)–(5) and its asymptotic behaviour as $k \to \infty$ is as in (14). Alternatively, $\Phi_2(x,t,k)$ can be represented in the form

$$\Phi_2(x,t,k) = \left(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int_{-t}^{t} L(x,t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right)\left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{-x}^{x} K(x,y,0)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right). \tag{16}$$

This form will be used for the study of the x-equation for $t = 0$. The kernels $K(x,y,t)$, $L_0(t,s) = L(0,t,s)$, $L(x,t,s)$ and $K(x,y,0)$ are absolutely continuous functions on their arguments.

Finally, the eigenfunction $\Phi_3(x,t,k)$ is normalized by the condition $\Phi_3(L,0,k) = \mathrm{e}^{\mathrm{i}kL\sigma_3}$ and has the representation

$$\Phi_3(x,t,k) = \left(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int_{-t}^{t} L(x,t,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s\right)\left(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{x}^{2L-x} K_L(x,y)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y\right). \tag{17}$$

The matrix $\Phi_3(x,t,k)$ possesses properties (1)–(5); moreover,

$$\Phi_3(x,t,k) = I + O(k^{-1}) + O(k^{-1}\,\mathrm{e}^{2\mathrm{i}k(x-L)\sigma_3}), \qquad \text{as} \quad k \to \infty. \tag{18}$$

Since all the introduced matrix-valued functions $\Phi_j(x,t,k)$ $(j=1,2,3)$, $\Phi_0(x,t,k)$ and $\Phi_T(x,t,k)$ are solutions of the x- and t-equations (4), (5), they are linear dependent, so there exist transition matrices $S(k)$, $S^T(k)$, $R(k)$ and $P(k)$ independent of x and t such that

$$\begin{aligned} \Phi_1(x,t,k) &= \Phi_2(x,t,k)S^T(k), \qquad & \Phi_2(x,t,k) &= \Phi_3(x,t,k)S(k), \\ \Phi_1(x,t,k) &= \Phi_3(x,t,k)R(k), \qquad & \Phi_T(x,t,k) &= \Phi_0(x,t,k)P(k). \end{aligned} \tag{19}$$

They can be written as follows:

$$\begin{aligned} S(k) &= \Phi_3^{-1}(0,0,k), \qquad & S^T(k) &= \Phi_1(0,0,k), \\ R(k) &= S(k)S^T(k), \qquad & P(k) &= \Phi_0^{-1}(0,0,k)\Phi_T(0,0,k). \end{aligned}$$

The integral representations for the eigenfunctions imply the following integral representations for the transition matrices:

$$S^{-1}(k) = \begin{pmatrix} \bar a(\bar k) & b(k) \\ -\bar b(\bar k) & a(k) \end{pmatrix} = I + \int_0^{2L} K_L(0,y)\,\mathrm{e}^{-\mathrm{i}ky\sigma_3}\,\mathrm{d}y, \tag{20}$$

$$S^T(k) = \begin{pmatrix} \bar A(\bar k) & B(k) \\ -\bar B(\bar k) & A(k) \end{pmatrix} = I + \frac{\mathrm{i}}{4k}\int_0^{2T} L_T(0,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s, \tag{21}$$

$$P(k) = \begin{pmatrix} \bar a_P(\bar k) & b_P(k) \\ -\bar b_P(\bar k) & a_P(k) \end{pmatrix} = I + \int_0^{2T} M_T(0,s)\,\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}\,\mathrm{d}s. \tag{22}$$

These formulae give a complete description of the transition matrices in terms of their Fourier transforms (in the last two cases, with respect to $\lambda = \frac{1}{4k}$): $K_L(0,y) \in H^1(0,L)$, $L_T(0,s) \in H^1(0,T)$, $M_T(0,s) \in L^2(0,T)$.

Remark 2.1. If Φ is a 2×2 matrix we denote its columns by Φ^-, Φ^+, i.e. $\Phi = (\Phi^-, \Phi^+)$.

3. Spectral problem for the *x*-equation

The basic scattering relation for the x-equation (4) has the form

$$\Phi_2(x,0,k) = \Phi_3(x,0,k)S(k), \tag{23}$$

where $S(k)$ is defined by the second formula in (19).

Let $u(x) \in H^1(0,L)$. Then, relation (23) defines a map

$$\mathbb{S} : \{u(x)\} \to \{a(k), b(k)\} \tag{24}$$

by the formula

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \Psi(0,k),$$

where the vector-function $\Psi(x,k) := \Phi_3^+(x,0,k)$ (the second column of the matrix $\Phi_3(x,t,k)$) satisfies the equation

$$\Psi_x + \mathrm{i}k\sigma_3\Psi = Q_0(x)\Psi, \qquad 0 < x < L, \tag{25}$$

and the boundary condition

$$\Psi(L,k)\,\mathrm{e}^{-\mathrm{i}kL} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix $Q_0(x)$ is defined by the initial function:

$$Q_0(x) = \begin{pmatrix} 0 & u(x) \\ -\bar u(x) & 0 \end{pmatrix}.$$

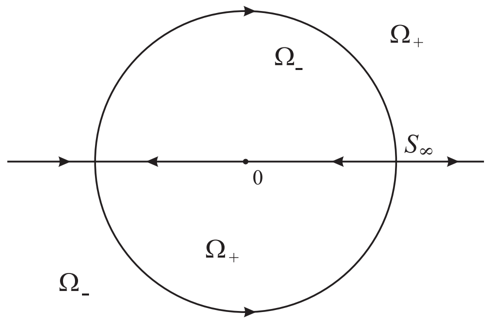

Figure 1. The contour Γ for the x-problem.

Properties of the spectral data $a(k)$ and $b(k)$:

(1) $a(k)$ and $b(k)$ are entire analytic functions of the exponential type $2L$ represented in the form

$$a(k) = 1 + \int_0^{2L} \alpha(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}y, \qquad b(k) = \int_0^{2L} \beta(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}y,$$

where $\alpha(y), \beta(y) \in H^1(0, L)$;

(2) $\det S(k) := a(k)\bar{a}(\bar{k}) + b(k)\bar{b}(\bar{k}) \equiv 1, k \in \mathbb{C}$;

(3) $a(k) = 1 + O(k^{-1}) + O(k^{-1}\,\mathrm{e}^{2\mathrm{i}kL}), b(k) = O(k^{-1}) + O(k^{-1}\,\mathrm{e}^{2\mathrm{i}kL}), k \to \infty$.

The map $\mathbb{Q}$, which is inverse to the map (24), can be written as follows:

$$u(x) = 2\mathrm{i} \lim_{k\to\infty} k M_{12}^{(x)}(x, k), \tag{26}$$

where $M_{12}^{(x)}(x, k)$ is (12) entry of the matrix $M^{(x)}(x, k)$ that is the solution of the following matrix Riemann–Hilbert problem (RH$_x$):

- $M^{(x)}(x, k) = \begin{cases} M_-^{(x)}(x,k), k\in\Omega_- \\ M_+^{(x)}(x,k), k\in\Omega_+ \end{cases}$ is a sectionally analytic matrix-valued function in $k \in \mathbb{C}\backslash\Gamma$, where the oriented contour Γ (figure 1) in the complex k-plane is a union of real line $\mathbb{R}$ and a circle $\mathcal{S}_\infty$: $\mathcal{S}_\infty = \{k \in \mathbb{C} : |k| = |\mathcal{S}_\infty|\}$, where $|\mathcal{S}_\infty|$ is a sufficiently large positive number. The orientation of Γ is chosen in such a way that k-plane is a union of the two open domains $\Omega_\pm$ and their common boundary Γ:

$$\mathbb{C} = \Omega_+ \cup \Omega_- \cup \Gamma, \qquad \Gamma = \mathbb{R} \cup \mathcal{S}_\infty,$$
$$\Omega_+ = \{k \in \mathbb{C} : |k| > |\mathcal{S}_\infty|, \mathrm{Im}\, k > 0\} \cup \{k \in \mathbb{C}, |k| < |\mathcal{S}_\infty|, \mathrm{Im}\, k < 0\},$$
$$\Omega_- = \{k \in \mathbb{C} : |k| > |\mathcal{S}_\infty|, \mathrm{Im}\, k < 0\} \cup \{k \in \mathbb{C}, |k| < |\mathcal{S}_\infty|, \mathrm{Im}\, k > 0\}.$$

- $\det M^{(x)}(x, k) \equiv 1$.
- $M_+^{(x)}(x, k) = M_-^{(x)}(x, k) J^{(x)}(x, k), k \in \Gamma$,
  where

$$J^{(x)}(x, k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{R}, \quad |k| < |\mathcal{S}_\infty| \\ \begin{pmatrix} 1 & \frac{b(k)}{a(k)}\,\mathrm{e}^{-2\mathrm{i}kx} \\ \frac{\bar{b}(\bar{k})}{\bar{a}(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kx} & \frac{1}{|a(k)|^2} \end{pmatrix}, & k \in \mathbb{R}, \quad |k| > |\mathcal{S}_\infty| \end{cases} \tag{27}$$

$$J^{(x)}(x,k)=\begin{cases}\begin{pmatrix}1 & \frac{b(k)}{a(k)}\,\mathrm{e}^{-2\mathrm{i}kx}\\ 0 & 1\end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \arg k\in(0,\pi),\\ \begin{pmatrix}1 & 0\\ \frac{\bar b(\bar k)}{\bar a(\bar k)}\,\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \arg k\in(\pi,2\pi);\end{cases} \tag{28}$$

- $M^{(x)}(x,k)=I+O(k^{-1}),\ k\to\infty$.
- $\mathbb{Q}:\{a(k),b(k)\}\to\{u(x)\}$ is inverse to $\mathbb{S}$.

Proof. Let the matrices $M_\pm^{(x)}(x,k)$ be as follows:

$$M_+^{(x)}(x,k)=\begin{cases}\left(\Phi_2^-(x,0,k)\,\mathrm{e}^{\mathrm{i}kx}, \quad \dfrac{\Phi_3^+(x,0,k)}{a(k)}\,\mathrm{e}^{-\mathrm{i}kx}\right), & k\in\mathbb{C}_+, \quad |k|>|\mathcal{S}_\infty|\\ \left(\Phi_2^-(x,0,k)\,\mathrm{e}^{\mathrm{i}kx}, \quad \Phi_2^+(x,0,k)\,\mathrm{e}^{-\mathrm{i}kx}\right), & k\in\mathbb{C}_-, \quad |k|<|\mathcal{S}_\infty|\end{cases}$$

$$M_-^{(x)}(x,k)=\begin{cases}\left(\Phi_2^-(x,0,k)\,\mathrm{e}^{\mathrm{i}kx}, \quad \Phi_2^+(x,0,k)\,\mathrm{e}^{-\mathrm{i}kx}\right), & k\in\mathbb{C}_+, \quad |k|<|\mathcal{S}_\infty|\\ \left(\dfrac{\Phi_3^-(x,0,k)}{\bar a(\bar k)}\,\mathrm{e}^{\mathrm{i}kx}, \quad \Phi_2^+(x,0,k)\,\mathrm{e}^{-\mathrm{i}kx}\right), & k\in\mathbb{C}_-, \quad |k|>|\mathcal{S}_\infty|\end{cases}$$

where $\Phi_2^-(x,0,k)$ and $\Phi_3^-(x,0,k)$ are the eigenfunctions evaluated at $t=0$. We choose the radius $|\mathcal{S}_\infty|$ of the circle $\mathcal{S}_\infty$ to be large enough so that $a(k)\neq 0$ for $\operatorname{Im}k\geqslant 0$ and $\bar a(\bar k)\neq 0$ for $\operatorname{Im}k\leqslant 0$ when $|k|>|\mathcal{S}_\infty|$. Then $M_+^{(x)}(x,k)$ is analytic in Ω_+ and $M_-^{(x)}(x,k)$ is analytic in Ω_-. For $k\in\mathbb{R}$ and $|k|<|\mathcal{S}_\infty|$, the jump matrix $J^{(x)}$ is trivial: $J^{(x)}(x,k)\equiv I$. For $k\in\mathbb{R}$ and $|k|>|\mathcal{S}_\infty|$ it is easy to see that jump matrix $J^{(x)}$ coincides with (27). For $|k|=|\mathcal{S}_\infty|$ and $0<\arg k<\pi$ it coincides with (28) as well as for $|k|=|\mathcal{S}_\infty|$ and $\pi<\arg k<2\pi$. The large-k asymptotic formulae for the eigenfunctions given in section 2 imply the following asymptotic expansion for the matrix $M^{(x)}(x,k)$:

$$M^{(x)}(x,k)=I+\frac{m^{(1)}(x)}{k}+O\left(\frac{1}{k^2}\right), \qquad k\to\infty.$$

Then, by (25), it follows that $u(x)$ is related to $M^{(x)}$ by

$$u(x)=2\mathrm{i}m_{12}^{(1)}(x)=2\mathrm{i}\lim_{k\to\infty}kM_{12}^{(x)}(x,k). \tag{29}$$

□

Now we show that relation (29) defines the map

$$\mathbb{Q}:\{a(k),b(k)\}\to\{u(x)\},$$

which is inverse to the spectral map (24):

$$\mathbb{S}:\{u(x)\}\to\{a(k),b(k)\}.$$

Consider the Riemann–Hilbert problem RH_x with the jump matrix $J^{(x)}(x,k)$ constructed by given spectral data $\{a(k),b(k)\}$. Then the following statement holds:

- The Riemann–Hilbert problem RH_x has a unique solution.
- The matrices $M_\pm^{(x)}(x,k)\,\mathrm{e}^{\mathrm{i}kx\sigma_3}$, $k\in\Omega_\pm$, satisfy the x-equation (25) with

$$Q_0(x)=\begin{pmatrix}0 & u_0(x)\\ -\bar u_0(x) & 0\end{pmatrix},$$

where $u_0(x)\in H^1(0,L)$.

- The spectral functions $a_0(k)$ and $b_0(k)$ defined by $u_0(x)$ via the direct map coincide with the spectral functions $a(k)$ and $b(k)$.

The proof of these statements follows the same lines as for the whole line case, see, e.g., [6, 7]. Since $a(k)$ and $b(k)$ are entire analytic functions of the exponential type $2L$, it follows that the support of the function $u_0(x)$ is the interval $[0, L]$. To prove that the spectral functions $a_0(k)$ and $b_0(k)$ coincide with, respectively, $a(k)$ and $b(k)$, we observe that the eigenfunction $\Psi_0(x, k)$ of the x-equation normalized by the condition $\Psi_0(L, k) = \mathrm{e}^{-\mathrm{i}kL\sigma_3}$ can be written in the form

$$\Psi_0(x, k) = M_+^{(x)}(x, k)\,\mathrm{e}^{\mathrm{i}kx\sigma_3}C_+(k),$$

where the matrix $C_+(k)$ is independent of x. Also observe that the Riemann–Hilbert problem $\mathrm{RH}_L(x = L)$ can be solved explicitly:

$$M_-^{(x)}(L, k) = \begin{cases} \begin{pmatrix} a(k) & -b(k)\,\mathrm{e}^{-2\mathrm{i}kL} \\ \bar{b}(\bar{k})\,\mathrm{e}^{2\mathrm{i}kL} & \bar{a}(\bar{k}) \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \qquad k \in \mathbb{C}_-, \\ \begin{pmatrix} \frac{1}{\bar{a}(\bar{k})} & -b(k)\,\mathrm{e}^{-2\mathrm{i}kL} \\ 0 & \bar{a}(\bar{k}) \end{pmatrix}, & |k| > |\mathcal{S}_\infty|, \qquad k \in \mathbb{C}_-; \end{cases}$$

$$M_+^{(x)}(L, k) = \begin{cases} \begin{pmatrix} a(k) & 0 \\ \bar{b}(\bar{k})\,\mathrm{e}^{2\mathrm{i}kL} & \frac{1}{a(k)} \end{pmatrix}, & |k| > |\mathcal{S}_\infty|, \qquad k \in \mathbb{C}_+, \\ \begin{pmatrix} a(k) & -b(k)\,\mathrm{e}^{-2\mathrm{i}kL} \\ \bar{b}(\bar{k})\,\mathrm{e}^{2\mathrm{i}kL} & \bar{a}(\bar{k}) \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \qquad k \in \mathbb{C}_+. \end{cases}$$

Therefore, since $\Psi_0(L, k)\,\mathrm{e}^{\mathrm{i}kL\sigma_3} = I$, it follows that

$$C_+(k) = \begin{cases} \begin{pmatrix} \frac{1}{a(k)} & 0 \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k = 0, \\ \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \quad \operatorname{Im} k = 0. \end{cases}$$

Hence, the spectral data,

$$S_0^{-1}(k) = \Psi_0^{-1}(0, k) := \begin{pmatrix} \bar{a_0}(\bar{k}) & b_0(k) \\ -\bar{b_0}(\bar{k}) & a_0(k) \end{pmatrix},$$

related to the function $u_0(x)$ (29) are defined by the equation

$$S_0^{-1}(k) = M_+^{(x)}(0, k)C_+(k).$$

If $x = 0$, the above RH problem $\mathrm{RH}_x|_{x=0}$ can also be solved explicitly. Indeed, since

$$J^{(x)}(0, k) = \begin{pmatrix} 1 & \frac{b(k)}{a(k)} \\ \frac{\bar{b}(\bar{k})}{\bar{a}(\bar{k})} & \frac{1}{|a(k)|^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\bar{b}(\bar{k})}{\bar{a}(\bar{k})} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b(k)}{a(k)} \\ 0 & 1 \end{pmatrix},$$

it follows that $M_+^{(x)}(0, k)$ takes the form

$$M_+^{(x)}(0, k) = \begin{pmatrix} 1 & \frac{b(k)}{a(k)} \\ 0 & 1 \end{pmatrix}, \qquad |k| > |\mathcal{S}_\infty|, \qquad \operatorname{Im} k > 0.$$

Therefore,

$$S_0^{-1}(k) = M_+^{(x)}(0, k)C_+(k) = \begin{pmatrix} \frac{1-|b(k)|^2}{a(k)} & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix} = S^{-1}(k),$$

i.e., $a_0(k) \equiv a(k)$ and $b_0(k) \equiv b(k)$ for $|k| > |\mathcal{S}_\infty|$ and $\operatorname{Im} k > 0$ and thus for all $k \in \mathbb{C}$.

4. Spectral problem for the *t*-equation

The basic scattering relation for the t-equation (5) for $x = 0$ has the form

$$\Phi_1(0,t,k) = \Phi_2(0,t,k)S^T(k). \tag{30}$$

Let $v(t), w(t) \in H^1(0,T)$ be such that $v^2(t) + |w^2(t)| \equiv 1$. Then, relation (30) defines the map

$$\mathbb{S}^T : \{v(t), w(t)\} \to \{A(k), B(k)\} \tag{31}$$

as follows:

$$\begin{pmatrix} B(k) \\ A(k) \end{pmatrix} = \hat{\Psi}^+(0,k),$$

where the vector-function $\hat{\Psi}^+(t,k) = \Phi_T^+(0,t,k)$ satisfies the equation

$$\hat{\Psi}_t^+ = \frac{\mathrm{i}}{4k}\hat{Q}(0,t)\hat{\Psi}^+, \qquad t \in (0,T), \tag{32}$$

with the coefficient matrix

$$\widehat{Q}(0,t) = \begin{pmatrix} v(t) & \mathrm{i}w(t) \\ -\mathrm{i}\bar{w}(t) & -v(t) \end{pmatrix}$$

and the initial condition: $\hat{\Psi}^+(T,k) = \mathrm{e}^{\frac{\mathrm{i}T\sigma_3}{4k}}$.

Properties of the spectral data $A(k)$ and $B(k)$ (t-problem)

(1) $A(k), B(k)$ are entire analytic functions of the exponential type $2T$ in the complex plane $\lambda = \frac{1}{4k}$; they can be represented in the form

$$A(k) = 1 + \frac{\mathrm{i}}{4k}\int_0^{2T} \hat{\alpha}(t)\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}}\,\mathrm{d}t, \qquad B(k) = \frac{\mathrm{i}}{4k}\int_0^{2T} \hat{\beta}(t)\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}}\,\mathrm{d}t, \tag{33}$$

where $\hat{\alpha}(t), \hat{\beta}(t) \in H^1(0,T)$;

(2) $\det S^T(k) := A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) \equiv 1$, $k \in \mathbb{C}\backslash\{0\}$;

(3) $A(k) = 1 + O(k^{-1})$, $B(k) = O(k^{-1})$, $k \to \infty$, $k \in \mathbb{C}$.

Now observe that as $k \to 0$,

$$A(k) = 1 - \hat{\alpha}(0) + \hat{\alpha}(2T)\,\mathrm{e}^{\frac{\mathrm{i}T}{4k}} + O(k);$$

hence, in general, there exists a sequence $k_j \in \mathbb{C}_- \cup \mathbb{R}$ such that $A(k_j) = 0$ and $\lim_{j\to\infty} k_j = 0$. This implies that the map inverse to $\mathbb{S}^T$ cannot be given in general in terms of a Riemann–Hilbert problem constructed directly in terms of $A(k)$ and $B(k)$ because of the singularities of $1/A(k)$ near $k = 0$, which can accumulate at $k = 0$. Note, however, that under the additional conditions ($v(0) = v(T) = -1$, $w(0) = w(T) = 0$) on the boundary data $v(t)$ and $w(t)$ one can prove that $A(k) = 1 + O(k)$, $k \to 0$ and $B(k) = O(k)$, $k \to 0$. In the special case when $A(k) \neq 0$ for all k, the inverse map can be constructed by the same way as in [2].

In order to construct the inverse map in the general case, we introduce the auxiliary spectral functions. Consider the solution $\Phi_0(0,t,k)$ of the t-equation normalized by the condition $\Phi_0(0,0,k) = m(0,0) = m_0$. The matrix $m_0 \in SU(2)$ is completely defined by the boundary data $v(t)$ and $w(t)$:

$$m_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-v(0)} & \frac{\mathrm{i}w(0)}{\sqrt{1-v(0)}} \\ \frac{\mathrm{i}\bar{w}(0)}{\sqrt{1-v(0)}} & \sqrt{1-v(0)} \end{pmatrix}. \tag{34}$$

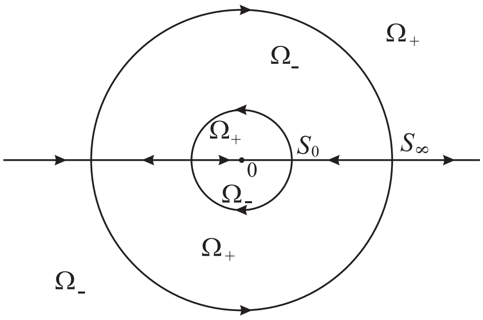

Figure 2. The contour Γ for t-problem.

Also consider the solution $\Phi_T(0,t,k)$ of the t-equation normalized by the condition $\Phi_T(0,T,k) = m_d(0,T)\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}})$. Let $P(k)$ be the transition matrix relating these solutions:

$$\Phi_T(0,t,k) = \Phi_0(0,t,k)P(k). \tag{35}$$

Since $\Phi_1(0,T,k) = \mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}$ and $\Phi_2(0,0,k) = I$, it follows that

$$P(k) = m^{-1}(0,0)S^T(k)\,\mathrm{e}^{\frac{\mathrm{i}T\sigma_3}{4k}}\,m_d(0,T)\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}}. \tag{36}$$

By (22) we have

$$P(k) = \begin{pmatrix} \bar{a}_P(\bar{k}) & b_P(k) \\ -\bar{b}_P(\bar{k}) & a_P(k) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(k) + O(k\,\mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{2k}}), \qquad k \to 0.$$

Summarizing, we have associated with the coefficient matrix $\hat{Q}(0,t)$ of the t-equation two sets of spectral functions: the spectral functions $A(k)$ and $B(k)$ with well-controlled asymptotic behaviour as $k \to \infty$ and the spectral functions $a_P(k)$ and $b_P(k)$ with well-controlled behaviour as $k \to 0$. These spectral functions are related by (36). It follows from (22) that $a_P(k)$ and $b_P(k)$ are entire analytic functions of $\lambda = \frac{1}{4k}$, of the exponential type $2T$, which can be written in the form

$$a_P(k) = 1 + \int_0^{2T} \tilde{\alpha}(t)\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}}\,\mathrm{d}t, \qquad b_P(k) = \int_0^{2T} \tilde{\beta}(t)\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}}\,\mathrm{d}t,$$

where $\tilde{\alpha}(t), \tilde{\beta}(t) \in L^2(0,T)$. Moreover,

$$a_P(k)\bar{a}_P(\bar{k}) + b_P(k)\bar{b}_P(\bar{k}) \equiv 1.$$

In order to formulate the Riemann–Hilbert problem RH_t, we define the contour Γ (figure 2) as follows:

$$\Gamma = \mathbb{R} \cup \mathcal{S}_0 \cup \mathcal{S}_\infty,$$
$$\mathcal{S}_0 = \{k \in \mathbb{C} : |k| = |\mathcal{S}_0| < 1\},$$
$$\mathcal{S}_\infty = \{k \in \mathbb{C} : |k| = |\mathcal{S}_\infty| > 1\},$$

where $|\mathcal{S}_0| > 0$ is sufficiently small so that $a_p(k) \neq 0$ for $|k| < |\mathcal{S}_0|$, $\operatorname{Im} k < 0$, and $|\mathcal{S}_\infty| > 0$ is sufficiently large so that $A(k) \neq 0$ for $|k| > |\mathcal{S}_\infty|$, $\operatorname{Im} k < 0$. The orientation of Γ is chosen in such a way that the k-plane is a union of two open domains $\Omega_\pm$:

$$\Omega_+ = \{k \in \mathbb{C} : |k| > |\mathcal{S}_\infty|, \operatorname{Im} k > 0\} \cup \{k \in \mathbb{C} : |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|, \operatorname{Im} k < 0\}$$
$$\cup \{k \in \mathbb{C} : |k| < |\mathcal{S}_0|, \operatorname{Im} k > 0\},$$
$$\Omega_- = \{k \in \mathbb{C} : |k| > |\mathcal{S}_\infty|, \operatorname{Im} k < 0\} \cup \{k \in \mathbb{C} : |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|, \operatorname{Im} k > 0\}$$
$$\cup \{k \in \mathbb{C} : |k| < |\mathcal{S}_0|, \operatorname{Im} k < 0\}.$$

The inverse (to (21)) map $\mathbb{Q}^T$ can be written as follows:

$$\hat{Q}(t) = -m(t)\sigma_3 m^{-1}(t),$$

where 2×2 matrix $m(t)$ is defined by the solution of the following Riemann–Hilbert problem RH_t:

- $M^{(t)}(t,k) = \begin{cases} M_-^{(t)}(t,k), & k\in\Omega_- \\ M_+^{(t)}(t,k), & k\in\Omega_+ \end{cases}$
  is sectionally analytic matrix-valued function in $k\in\mathbb{C}\backslash\Gamma$.
- $\det M^{(t)}(t,k)\equiv 1$.
- $M_+^{(t)}(t,k) = M_-^{(t)}(t,k)J^{(t)}(t,k)$, $k\in\Gamma$, where

$$J^{(t)}(t,k) = \begin{cases} \begin{pmatrix} \frac{1}{|A(k)|^2} & -\frac{B(k)}{A(k)}\,\mathrm{e}^{-\frac{\mathrm{i}t}{2k}} \\ -\frac{\bar B(\bar k)}{\bar A(\bar k)}\,\mathrm{e}^{\frac{\mathrm{i}t}{2k}} & 1 \end{pmatrix}, & k\in\mathbb{R}, \quad |k|>|\mathcal{S}_\infty|, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k\in\mathbb{R}, \quad |\mathcal{S}_0|<|k|<|\mathcal{S}_\infty|, \\ \begin{pmatrix} \frac{1}{|a_P(k)|^2} & -\frac{b_P(k)}{a_P(k)}\,\mathrm{e}^{-\frac{\mathrm{i}t}{2k}} \\ -\frac{\bar b_P(\bar k)}{\bar a_P(\bar k)}\,\mathrm{e}^{\frac{\mathrm{i}t}{2k}} & 1 \end{pmatrix}, & k\in\mathbb{R}, \quad |k|<|\mathcal{S}_0|, \end{cases}$$

$$J^{(t)}(t,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{\bar B(\bar k)}{\bar A(\bar k)}\,\mathrm{e}^{\frac{\mathrm{i}t}{2k}} & 1 \end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \operatorname{Im} k>0, \\ \begin{pmatrix} 1 & -\frac{B(k)}{A(k)}\,\mathrm{e}^{-\frac{\mathrm{i}t}{2k}} \\ 0 & 1 \end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \operatorname{Im} k<0, \end{cases}$$

$$J^{(t)}(t,k) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}t}{4k}\sigma_3} m_0 \begin{pmatrix} 1 & 0 \\ -\frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}t}{4k}\sigma_3}, & |k|=|\mathcal{S}_0|, \quad \operatorname{Im} k>0, \\ \mathrm{e}^{-\frac{\mathrm{i}t}{4k}\sigma_3} \begin{pmatrix} 1 & -\frac{b_P(k)}{a_P(k)} \\ 0 & 1 \end{pmatrix} m_0^{-1}\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}\sigma_3}, & |k|=|\mathcal{S}_0|, \quad \operatorname{Im} k<0, \end{cases}$$

  where m_0 is given by (34).

- $M_\pm^{(t)}(t,k) = I + O(k^{-1})$, $k\to\infty$.
- $M_\pm^{(t)}(t,k) = m_\pm(t) + O(k)$, $k\to 0$, $k\in\Omega_\pm$, where $m_+(t) = m_-(t) = m(t)$ is a unitary 2×2 matrix.
- $\mathbb{Q}^T$ is inverse to $\mathbb{S}^T$ (21).

Proof. Let matrices $M_\pm^{(t)}(t,k)$ be as follows:

$$M_+^{(t)}(t,k) = \begin{cases} \left(\frac{\Phi_1^-(0,t,k)}{\bar A(\bar k)}, \quad \Phi_2^+(0,t,k)\right)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, & |k|>|\mathcal{S}_\infty|, \quad \operatorname{Im} k>0, \\ \left(\frac{\Phi_T^-(0,t,k)}{\bar a_P(\bar k)}, \quad \Phi_0^+(0,t,k)\right)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, & |k|<|\mathcal{S}_0|, \quad \operatorname{Im} k>0, \end{cases}$$

$$M_+^{(t)}(t,k) = M_-^{(t)}(t,k) = \left(\Phi_2^-(0,t,k), \quad \Phi_2^+(0,t,k)\right)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, \quad |\mathcal{S}_0|<|k|<|\mathcal{S}_\infty|,$$

$$M_-^{(t)}(t,k) = \begin{cases} \left(\Phi_2^-(0,t,k), \quad \frac{\Phi_1^+(0,t,k)}{A(k)}\right)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, & |k|>|\mathcal{S}_\infty|, \quad \operatorname{Im} k<0, \\ \left(\Phi_0^-(0,t,k), \quad \frac{\Phi_T^+(0,t,k)}{a_P(k)}\right)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, & |k|<|\mathcal{S}_0|, \quad \operatorname{Im} k<0. \end{cases}$$

□

The radius of the circle $|\mathcal{S}_\infty|$ ($|\mathcal{S}_0|$) is large (small) enough so that $A(k) \neq 0$ for $|k| > |\mathcal{S}_\infty|$, $\operatorname{Im} k < 0$ ($a_P(k) \neq 0$ for $|k| < |\mathcal{S}_0|, \operatorname{Im} k < 0$). Then $M_\pm^{(t)}(t,k)$ is analytic in $\Omega_\pm$. The scattering relation (30)

$$\frac{\Phi_1^-(0,t,k)}{\bar{A}(\bar{k})} = \Phi_2^-(0,t,k) - \frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\Phi_2^+(0,t,k), \qquad k \in \mathbb{C}_+ \cup \mathbb{R},$$
$$\frac{\Phi_1^+(0,t,k)}{A(k)} = \frac{B(k)}{A(k)}\Phi_2^-(0,t,k) + \Phi_2^+(0,t,k), \qquad k \in \mathbb{C}_- \cup \mathbb{R},$$

and relation (35)

$$\frac{\Phi_T^-(0,t,k)}{\bar{a}_P(\bar{k})} = \Phi_0^-(0,t,k) - \frac{\bar{b}_P(\bar{k})}{\bar{a}_P(\bar{k})}\Phi_0^+(0,t,k),$$
$$\frac{\Phi_T^+(0,t,k)}{a_P(k)} = \frac{b_P(k)}{a_P(k)}\Phi_0^-(0,t,k) + \Phi_0^+(0,t,k)$$

written in the vector form imply that $\det M_-^{(t)}(t,k) = \det M_+^{(t)}(t,k) \equiv 1$. These relations define the jump matrix $J^{(t)}(t,k)$ for $k \in \Gamma\backslash\mathcal{S}_0$. The jump matrix $J^{(t)}(t,k)$ for $|k| = |\mathcal{S}_0|$ emerges from the equation

$$\Phi_0(0,t,k) = \Phi_2(0,t,k)m(0,0) = \Phi_2(0,t,k)m_0.$$

The asymptotic behaviour of $M_\pm^{(t)}(t,k)$ (as $k \to \infty$ and as $k \to 0$) follows from the asymptotic formulae for the corresponding vector functions (see section 2) and the asymptotic behaviour of the spectral functions $A(k)$, $\bar{A}(\bar{k})$, $a_P(k)$ and $\bar{a}_P(\bar{k})$. The matrices $m_\pm(t)$ are such that $m_+(t) = m_-(t) = m_d(0,t) = m(0,t)d^{\sigma_3}(0,t) \in SU(2)$. Thus, the coefficient matrix $\hat{Q}(0,t)$ can be reconstructed by the formula

$$\hat{Q}(t) = -m_-(t)\sigma_3 m_-^{-1}(t) = -m_+(t)\sigma_3 m_+^{-1}(t) = -m(t)\sigma_3 m^{-1}(t). \tag{37}$$

Now we show that formula (37) defines the map

$$\mathbb{Q}^T : \{A(k), B(k)\} \to \{v(t), w(t)\},$$

which is inverse to the spectral map (21):

$$\mathbb{S}^T : \{v(t), w(t)\} \to \{A(k), B(k)\}.$$

The auxiliary spectral data $a_P(k)$ and $b_P(k)$ are given in terms of $A(k)$ and $B(k)$ by (36).

Following the general ideas of [6] and using the results of [7] for contours with self-intersections, one can establish the following:

- The Riemann–Hilbert problem RH_t has a unique solution.
- The matrices $M_\pm^{(t)}(t,k)\,\mathrm{e}^{-\frac{it\sigma_3}{4k}}$, $k \in \Omega_\pm$, satisfy the t-equation (5) with
$$\hat{Q}(t) = -m(t)\sigma_3 m^{-1}(t).$$
- The matrix $\hat{Q}(t)$ is Hermitian, $Sp \quad \hat{Q}(t) = 0$, $\hat{Q}^2(t) = I$ and $\hat{Q}(t) \in H^1(0,T)$.
- The spectral functions $A_0(k)$ and $B_0(k)$ defined by the t-equation with the coefficient matrix $\hat{Q}(t)$ coincide with the functions $A(k)$ and $B(k)$, respectively, i.e.
$$A_0(k) \equiv A(k) \qquad B_0(k) \equiv B(k).$$

The proof of the first three statements follows the same lines as in [6] and [7]. To prove the last statement, we observe that solution $\hat{\Phi}(t,k)$ of the t-equation normalized by the condition $\hat{\Phi}(T,k) = \mathrm{e}^{-\frac{iT\sigma_3}{4k}}$ is related to $M_+^{(t)}(t,k)$ by the equation

$$\hat{\Phi}(t,k) = M_+^{(t)}(t,k)\,\mathrm{e}^{-\frac{it\sigma_3}{4k}}D_+(k), \qquad k \in \Omega_+.$$

For $t = T$, the Riemann–Hilbert problem RH_t is explicitly solved. In particular, we have

$$M_+^{(t)}(T,k) = \begin{pmatrix} \frac{1}{\bar A(\bar k)} & -B(k)\,\mathrm{e}^{-\frac{\mathrm{i}T}{2k}} \\ 0 & \bar A(\bar k) \end{pmatrix}, \qquad |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0.$$

Therefore,

$$D_+(k) = \mathrm{e}^{\frac{\mathrm{i}T\sigma_3}{4k}} \left[M_+^{(t)}(T,k)\right]^{-1} \mathrm{e}^{-\frac{\mathrm{i}T\sigma_3}{4k}} = \begin{pmatrix} \bar A(\bar k) & B(k) \\ 0 & \frac{1}{\bar A(\bar k)} \end{pmatrix}.$$

Since $S_0^T(k) = \hat\Phi(0,k)$, it follows that

$$S_0^T(k) = M_+^{(t)}(0,k) D_+(k).$$

For $t = 0$, the Riemann–Hilbert problem RH_t is also explicitly solved:

$$M_+^{(t)}(0,k) = \begin{pmatrix} 1 & 0 \\ -\frac{\bar B(\bar k)}{\bar A(\bar k)} & 1 \end{pmatrix}, \qquad |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0.$$

Therefore,

$$S_0^T(k) = \begin{pmatrix} 1 & 0 \\ -\frac{\bar B(\bar k)}{\bar A(\bar k)} & 1 \end{pmatrix} \begin{pmatrix} \bar A(\bar k) & B(k) \\ 0 & \frac{1}{\bar A(\bar k)} \end{pmatrix} = \begin{pmatrix} \bar A(\bar k) & B(k) \\ -\bar B(\bar k) & A(k) \end{pmatrix} = S^T(k),$$

i.e., $A_0(k) = A(k)$ and $B_0(k) = B(k)$ for $|k| > |\mathcal{S}_\infty|$, $\operatorname{Im} k > 0$ and thus for all $k \in \mathbb{C}\backslash\{0\}$.

5. Inverse problem for the compatible *x*- and *t*-equations: reconstruction of the SRS model

In this section, we give a reconstruction for the solution of the SRS equations in terms of the spectral functions $(a(k), b(k))$ and $(A(k), B(k))$ associated with the initial and boundary conditions. Under the assumption that x- and t-equations (4), (5) are fulfilled, relations (19) can be written in the form of a matrix Riemann–Hilbert problem.

Let $q(x,t)$, $\mu(x,t)$, $\nu(x,t)$ be absolutely continuous functions with respect to $x \in [0, L]$ and $t \in [0, T]$ satisfying the SRS equations (1), the initial conditions (2) and the boundary conditions (3). Then relations (19) define a map

$$\mathbb{S}^R : \{q(x,t), \nu(x,t), \mu(x,t)\} \to \{a(k), b(k), A(k), B(k)\}. \tag{38}$$

To formulate the Riemann–Hilbert problem, we use the spectral functions $\{a(k), b(k), A(k), B(k)\}$, the auxiliary spectral functions $a_P(k), b_P(k)$, which are the entries of the transition matrix $P(k)$ (22), and the auxiliary spectral functions $a_R(k), b_R(k)$, which are the entries of the transition matrix $R(k) = S(k)S^T(k)$. The matrix $P(k)$ is also defined by $S^T(k)$ (36). In the particular case when $v(0) = v(T) = -1$, we have $P(k) = S^T(k)d^{\sigma_3}(0,T)$.

The auxiliary spectral data have the following properties:

(1) $a_R(k), b_R(k), a_P(k), b_P(k)$ are analytic functions for $k \in \mathbb{C}\backslash\{0\}$;
(2) $a_R(k)\bar a_R(\bar k) + b_R(k)\bar b_R(\bar k) \equiv 1, a_P(k)\bar a_P(\bar k) + b_P(k)\bar b_P(\bar k) \equiv 1, k \in \mathbb{C}\backslash\{0\}$;
(3) $a_R(k) = 1 + O(k^{-1}), k \to \infty, \operatorname{Im} k \leqslant 0, b_R(k) = O(k^{-1}), k \to \infty, \operatorname{Im} k = 0; a_P(k) = 1 + O(k), b_P(k) = O(k), k \to 0, \operatorname{Im} k \leqslant 0$.

The inverse (to (38)) map $\mathbb{Q}^R$ is defined by

$$q(x,t) = 2\mathrm{i} \lim_{k\to\infty} k M_{12}(x,t,k), \tag{39}$$

$$\nu(x,t) = 1 - 2|m_{11}(x,t)|^2, \tag{40}$$

$$\mu(x,t) = -2\mathrm{i} m_{11}(x,t) m_{12}(x,t), \tag{41}$$

where $m_{ij}(x,t)$ are the entries of the matrix

$$m(x,t) = \lim_{k\to 0} M(x,t,k),$$

and $M(x,t,k)$ is the solution of the following Riemann–Hilbert problem RH$_{xt}$:

- $M(x,t,k) = \left\{\begin{smallmatrix} M_-(x,t,k), & k\in\Omega_- \\ M_+(x,t,k), & k\in\Omega_+ \end{smallmatrix}\right.$ is sectionally analytic for $k \in \mathbb{C}\backslash\Gamma$, where the domains $\Omega_\pm$ and the oriented contour Γ (figure 2) are the same as in the problem RH$_t$;
- $\det M(x,t,k) \equiv 1$ for $k \in \mathbb{C}\backslash\Gamma$;
- $M_+(x,t,k) = M_-(x,t,k) J(x,t,k)$, $k \in \Gamma$, where

$$J(x,t,k) = \begin{cases} \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} 1 & 0 \\ -\frac{\bar b_R(\bar k)}{\bar a_R(\bar k)} & 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| = |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0; \\ \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} \frac{1}{|a_R(k)|^2} & -\frac{b_R(k)}{a_R(k)} \\ -\frac{\bar b_R(\bar k)}{\bar a_R(\bar k)} & 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k = 0; \\ \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} 1 & -\frac{b_R(k)}{a_R(k)} \\ 0 & 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| = |\mathcal{S}_\infty|, \quad \operatorname{Im} k < 0; \end{cases}$$

$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|, \quad \operatorname{Im} k = 0;$$

$$J(x,t,k) = \begin{cases} \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} a(k) & -b(k) \\ \bar b(\bar k) & \bar a(\bar k) \end{pmatrix} m_0 \begin{pmatrix} 1 & 0 \\ -\frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| = |\mathcal{S}_0|, \operatorname{Im} k > 0, \\ \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} \frac{1}{|a_P(k)|^2} & -\frac{b_P(k)}{a_P(k)} \\ -\frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| < |\mathcal{S}_0|, \operatorname{Im} k = 0; \\ \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix} 1 & -\frac{b_P(k)}{a_P(k)} \\ 0 & 1 \end{pmatrix} m_0^{-1} \begin{pmatrix} \bar a(\bar k) & b(k) \\ -\bar b(\bar k) & a(k) \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta}, & |k| = |\mathcal{S}_0|, \operatorname{Im} k < 0, \end{cases}$$

  where $\theta = (kx + \frac{t}{4k})\sigma_3$ and m_0 is defined by (34);
- $M(x,t,k) = I + O(k^{-1})$, $k \to \infty$;
- $M_\pm(x,t,k) = m_\pm(x,t) + O(k)$, $k \to 0$, where $m_+(x,t) = m_-(x,t) = m(x,t)$ is a unitary 2×2 matrix;
- $\mathbb{Q}^R$ is inverse to $\mathbb{S}^R$ (38).

Proof. In order to construct the Riemann–Hilbert problem RH$_{xt}$, we define the following matrices:

$$M_+(x,t,k) = \begin{cases} \left(\frac{\Phi_1^-(x,t,k)}{\bar a_R(\bar k)}, \quad \Phi_3^+(x,t,k)\right)\mathrm{e}^{\mathrm{i}\theta}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0, \\ \left(\frac{\Phi_T^-(x,t,k)}{\bar a_P(\bar k)}, \quad \Phi_0^+(x,t,k)\right)\mathrm{e}^{\mathrm{i}\theta}, & |k| < |\mathcal{S}_0|, \quad \operatorname{Im} k > 0, \end{cases}$$

$$M_+(x,t,k) = M_-(x,t,k) = \Phi_3(x,t,k)\,\mathrm{e}^{\mathrm{i}\theta}, \qquad |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|,$$

$$M_-(x,t,k) = \begin{cases} \left(\Phi_3(x,t,k), \quad \frac{\Phi_1^+(x,t,k)}{a_R(k)}\right)\mathrm{e}^{\mathrm{i}\theta}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k < 0, \\ \left(\Phi_0^-(x,t,k), \quad \frac{\Phi_T^+(x,t,k)}{a_P(k)}\right)\mathrm{e}^{\mathrm{i}\theta}, & |k| < |\mathcal{S}_0|, \quad \operatorname{Im} k < 0, \end{cases}$$

where $\Phi_1^{\mp}(x,t,k)$, $\Phi_3^{\mp}(x,t,k)$, $\Phi_0^{\mp}(x,t,k)$, $\Phi_T^{\mp}(x,t,k)$ are the vector columns of the matrices $\Phi_1(x,t,k)$, $\Phi_3(x,t,k)$, $\Phi_0(x,t,k)$, $\Phi_T(x,t,k)$. The radius $|\mathcal{S}_\infty|$ ($|\mathcal{S}_0|$) of the circle $\mathcal{S}_\infty$ ($\mathcal{S}_0$) is sufficiently large (small) so that $a_R(k) \neq 0$ for $|k| \geqslant |\mathcal{S}_\infty|$, $\operatorname{Im} k < 0$ ($a_P(k) \neq 0$ for $|k| \leqslant |\mathcal{S}_0|$, $\operatorname{Im} k < 0$). Then, the matrices $M_\pm(x,t,k)$ are analytic functions in the domains $\Omega_\pm$. The determinants of these matrices are equal to 1, which follows from the vector relations

$$\begin{aligned}
\Phi_1^-(x,t,k) &= \bar{a}_R(\bar{k})\Phi_3^-(x,t,k) - \bar{b}_R(\bar{k})\Phi_3^+(x,t,k),\\
\Phi_1^+(x,t,k) &= b_R(k)\Phi_3^-(x,t,k) + a_R(k)\Phi_3^+(x,t,k),\\
\Phi_T^-(x,t,k) &= \bar{a}_P(\bar{k})\Phi_0^-(x,t,k) - b_P(k)\Phi_0^+(x,t,k),\\
\Phi_T^+(x,t,k) &= b_P(k)\Phi_0^-(x,t,k) + a_P(k)\Phi_0^+(x,t,k),
\end{aligned}$$

arising from (19). Using these relations and

$$\Phi_0(x,t,k) = \Phi_3(x,t,k)S(k)m_0,$$

the direct calculation gives the form of the jump matrix $J(x,t,k)$ on the different parts of Γ. The asymptotic formulae for $M(x,t,k)$ as $k \to \infty$ and $k \to 0$ follow from the corresponding equations for the eigenfunctions, see section 2, and from the asymptotic behaviour of the spectral functions $a_R(k)$ and $a_P(k)$. Note that

$$\begin{aligned}
M_-(x,t,k) &= m_d(x,t) + O(k), &\quad k \to 0, &\quad \operatorname{Im} k \leqslant 0,\\
M_+(x,t,k) &= m_d(x,t) + O(k), &\quad k \to 0, &\quad \operatorname{Im} k \geqslant 0,
\end{aligned}$$

where $m_d(x,t)$ is the same unitary matrix (8).

As for the t-problem, the general ideas of [6] and the results of [7] for contours with self-intersections imply the following statements:

- The Riemann–Hilbert problem RH_{xt} has a unique solution.
- The matrices $M_\pm(x,t,k)\,\mathrm{e}^{-\mathrm{i}kx\sigma_3-\frac{\mathrm{i}t\sigma_3}{4k}}$ for $k \in \Omega_\pm$ are absolutely continuous in x and t and satisfy x- and t-equations with matrices $Q(x,t)$ and $\hat{Q}(x,t)$, respectively, defined by (39)–(41).
- $q(x,t)$, $\nu(x,t)$, $\mu(x,t)$ are absolutely continuous and satisfy the SRS equations.
- The initial and boundary conditions are fulfilled, i.e. $q(x,0) = u(x)$, $\nu(0,t) = v(t)$, $\mu(0,t) = w(t)$.
- The spectral functions $a_0(k)$, $b_0(k)$, $A_0(k)$ and $B_0(k)$ defined by $q(x,t)$, $\nu(x,t)$ and $\mu(x,t)$ via the direct spectral map coincide with $a(k)$, $b(k)$, $A(k)$ and $B(k)$, respectively.

The first three statements can be proved by the same scheme as in [6] with the corresponding generalizations given in [7]. The fifth statement follows from the fourth one and literally repeats the proofs of the corresponding statements in sections 3 and 4. To prove the forth statement, we show that the Riemann–Hilbert problem RH_{xt} for $t = 0$ is equivalent to the Riemann–Hilbert problem RH_x in the following sense: there exists a matrix $G(x,k)$, sectionally analytic in k, such as

$$M^{(x)}(x,k) = M(x,0,k)G(x,k)$$

and

$$G(x,k) = I + \frac{D^1}{k} + \frac{D^2}{k^2} + \cdots + O\left(\frac{\mathrm{e}^{-C(k)x}}{k}\right), \qquad k \to \infty,$$

where $D^1, D^2, \ldots$ are constant diagonal matrices and $C(k)$ is a positive function. Indeed, define $\hat{M}(x,k)$ as follows:

$$\hat{M}(x,k) = \begin{cases} M_+(x,0,k)G_+(x,k), & k \in \Omega_+,\\ M_-(x,0,k)G_-(x,k), & k \in \Omega_-,\end{cases}$$

where

$$G_+(x,k) = G_-(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3} S(k)\,\mathrm{e}^{\mathrm{i}kx\sigma_3}, \qquad |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|,$$

$$G_+(x,k) = \mathrm{e}^{-2\mathrm{i}kx\sigma_3}\begin{pmatrix} 1 & 0 \\ \frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix} m_0^{-1}\,\mathrm{e}^{\mathrm{i}kx\sigma_3}, \qquad |k| < |\mathcal{S}_0|, \quad \operatorname{Im} k > 0,$$

$$G_-(x,k) = \mathrm{e}^{-2\mathrm{i}kx\sigma_3}\begin{pmatrix} 1 & -\frac{b_P(k)}{a_P(k)} \\ 0 & 1 \end{pmatrix} m_0^{-1}\,\mathrm{e}^{\mathrm{i}kx\sigma_3}, \qquad |k| < |\mathcal{S}_0|, \quad \operatorname{Im} k < 0,$$

$$G_+(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\begin{pmatrix} a(k) & 0 \\ \frac{\bar B(\bar k)}{\bar a_R(\bar k)} & \frac{1}{a(k)} \end{pmatrix} \mathrm{e}^{\mathrm{i}kx\sigma_3}, \qquad |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0,$$

$$G_-(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\begin{pmatrix} \frac{1}{\bar a(\bar k)} & -\frac{B(k)}{\bar a_R(\bar k)} \\ 0 & \bar a(\bar k) \end{pmatrix} \mathrm{e}^{\mathrm{i}kx\sigma_3}, \qquad |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k < 0.$$

Then the matrix $\hat M(x,k)$ is analytic in $\Omega_\pm$, and the jump matrix $\hat J(x,k) := \hat M_-^{-1}(x,k)\hat M_+(x,k)(k \in \Gamma)$ takes the form

$$\hat J(x,k) = G_-^{-1}(x,k) M_-^{-1}(x,0,k) M_+(x,0,k) G_+(x,k) = G_-^{-1}(x,k) J(x,0,k) G_+(x,k).$$

It is easy to verify that for $k \in \mathcal{S}_0$,

$$\hat J(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3} S^{-1}(k) S(k) m_0 \begin{pmatrix} 1 & 0 \\ -\frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\bar b_P(\bar k)}{\bar a_P(\bar k)} & 1 \end{pmatrix} m_0^{-1}\,\mathrm{e}^{\mathrm{i}kx\sigma_3} = I$$

for $\operatorname{Im} k > 0$ and

$$\hat J(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3} m_0 \begin{pmatrix} 1 & \frac{b_P(k)}{a_P(k)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{b_P(k)}{a_P(k)} \\ 0 & 1 \end{pmatrix} m_0^{-1} S^{-1}(k) S(k)\,\mathrm{e}^{\mathrm{i}kx\sigma_3} = I$$

for $\operatorname{Im} k < 0$. For $k \in \mathcal{S}_\infty$, we have

$$\hat J(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\begin{pmatrix} \bar a(\bar k) & b(k) \\ -\bar b(\bar k) & a(k) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{\bar b_R(\bar k)}{\bar a_R(\bar k)} & 1 \end{pmatrix}\begin{pmatrix} a(k) & 0 \\ \frac{\bar B(\bar k)}{\bar a_R(\bar k)} & \frac{1}{a(k)} \end{pmatrix}\mathrm{e}^{\mathrm{i}kx\sigma_3}$$
$$= \begin{pmatrix} 1 & \frac{b(k)}{a(k)}\,\mathrm{e}^{-2\mathrm{i}kx} \\ 0 & 1 \end{pmatrix}$$

for $\operatorname{Im} k > 0$ and

$$\hat J(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\begin{pmatrix} \bar a(\bar k) & \frac{B(k)}{a_R(k)} \\ 0 & \frac{1}{\bar a(\bar k)} \end{pmatrix}\begin{pmatrix} 1 & -\frac{b_R(k)}{a_R(k)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a(k) & -b(k) \\ \bar b(\bar k) & \bar a(\bar k) \end{pmatrix}\mathrm{e}^{\mathrm{i}kx\sigma_3}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{\bar b(\bar k)}{\bar a(\bar k)}\,\mathrm{e}^{2\mathrm{i}kx} & 1 \end{pmatrix}$$

for $\operatorname{Im} k < 0$.

Finally, for $k \in \mathbb{R}$ and $|k| > |\mathcal{S}_\infty|$

$$\hat J(x,k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\begin{pmatrix} \bar a(\bar k) & \frac{B(k)}{a_R(k)} \\ 0 & \frac{1}{\bar a(\bar k)} \end{pmatrix}\begin{pmatrix} \frac{1}{|a_R(k)|^2} & -\frac{b_R(k)}{a_R(k)} \\ -\frac{\bar b_R(\bar k)}{\bar a_R(\bar k)} & 1 \end{pmatrix}\begin{pmatrix} a(k) & 0 \\ \frac{\bar B(\bar k)}{\bar a_R(\bar k)} & \frac{1}{a(k)} \end{pmatrix}\mathrm{e}^{\mathrm{i}kx\sigma_3}$$
$$= \begin{pmatrix} 1 & \frac{b(k)}{a(k)}\,\mathrm{e}^{-2\mathrm{i}kx} \\ \frac{\bar b(\bar k)}{\bar a(\bar k)}\,\mathrm{e}^{2\mathrm{i}kx} & \frac{1}{|a(k)|^2} \end{pmatrix}.$$

Since the matrix $\hat{M}(x,k)$ is continuous across $k \in \mathcal{S}_0$ ($\hat{J}(x,k) = I, k \in \mathcal{S}_0$) and thus is analytic everywhere except on $\mathcal{S}_\infty$ and $(-\infty, -|\mathcal{S}_\infty|) \cup (|\mathcal{S}_\infty|, \infty)$, where $\hat{J}(x,k) \equiv J^{(x)}(x,k)$ according to (27), we conclude that $\hat{M}(x,k) \equiv M^{(x)}(x,k)$.

Now we show that $q(x,0) = u(x)$. Indeed, the following expansions

$$M(x,t,k) = I + \frac{M^1(x,t)}{2\mathrm{i}k} + O(k^{-2}), \qquad k \to \infty$$

$$M^{(x)}(x,k) = \hat{M}(x,k) = I + \frac{\hat{M}^1(x)}{2\mathrm{i}k} + O(k^{-2}), \qquad k \to \infty$$

give the equations

$$q(x,t) = M^1_{12}(x,t), \qquad u(x) = \hat{M}^1_{12}(x). \tag{42}$$

Since $M^{(x)}(x,k) = M(x,0,k)G(x,k)(k \in \Omega_+)$ and

$$G(x,k) = I + \frac{D^1}{k} + \frac{D^2}{k^2} + \cdots + O\left(\frac{\mathrm{e}^{-2\,\mathrm{Im}\,kx}}{|k|}\right), \qquad k \to \infty \quad (\mathrm{Im}\,k > 0),$$

where $D^1, D^2, \ldots$ are diagonal constant matrices, we find

$$\hat{M}^1(x) = M^1(x,0) + 2\mathrm{i}D^1.$$

Then, equations (42) yield

$$u(x) = \hat{M}^1_{12}(x) = M^1_{12}(x,0) = q(x,0).$$

Thus, the problem $\mathrm{RH}_{xt}|_{t=0}$ is equivalent to the problem RH_x.

The proof of the equivalence between the Riemann–Hilbert problem $\mathrm{RH}_{xt}|_{x=0}$ and the Riemann–Hilbert problem RH_t is as follows. Let

$$N_\pm(t,k) = M_\pm(0,t,k)H_\pm(t,k),$$

where

$$H_+(t,k) = H_-(t,k) = I, \qquad |k| < |\mathcal{S}_0|;$$

$$H_+(t,k) = H_-(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} S(k)\, \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}, \qquad |\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|;$$

$$H_+(t,k) = \begin{pmatrix} \frac{\bar{a}_R(\bar{k})}{\bar{A}(k)} & -b(k)\,\mathrm{e}^{-\frac{\mathrm{i}t}{2k}} \\ 0 & \frac{\bar{A}(\bar{k})}{\bar{a}_R(\bar{k})} \end{pmatrix}, \qquad |k| > |\mathcal{S}_\infty|, \quad \mathrm{Im}\,k > 0,$$

$$H_-(t,k) = \begin{pmatrix} \frac{A(k)}{a_R(k)} & 0 \\ \bar{b}(\bar{k})\,\mathrm{e}^{\frac{\mathrm{i}t}{2k}} & \frac{a_R(k)}{A(k)} \end{pmatrix}, \qquad |k| > |\mathcal{S}_\infty|, \quad \mathrm{Im}\,k < 0.$$

The matrices $N_\pm(t,k)$ are analytic in $\Omega_\pm$ and corresponding jump matrix $\tilde{J}(t,k) := N_-^{-1}(t,k)N_+(t,k)$ takes the form

$\tilde{J}(t,k) = H_-^{-1}(t,k)M_-^{-1}(0,t,k)M_+(0,t,k)H_+(t,k) = H_-^{-1}(t,k)J(0,t,k)H_+(t,k)$.

As above, it is easy to verify that $\tilde{J}(t,k) = J^{(t)}(t,k)$ for $k \in \mathcal{S}_0$ and for $\{|k| < |\mathcal{S}_0|\} \cap \{\mathrm{Im}\,k = 0\}$. For $\{|\mathcal{S}_0| < |k| < |\mathcal{S}_\infty|\} \cap \{\mathrm{Im}\,k = 0\}$,

$$\tilde{J}(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} S^{-1}(k)\, \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}\, I\, \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} S(k)\, \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}} = I.$$

For $|k| = |\mathcal{S}_\infty|$, we find

$$\tilde{J}(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} \begin{pmatrix} \frac{a_R(k)}{A(k)} & 0 \\ -\bar{b}(\bar{k}) & \frac{A(k)}{a_R(k)} \end{pmatrix} \begin{pmatrix} 1 & -\frac{b_R(k)}{a_R(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(k) & -b(k) \\ \bar{b}(\bar{k}) & \bar{a}(\bar{k}) \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}$$

$$= \begin{pmatrix} 1 & -\frac{B(k)}{A(k)}\,\mathrm{e}^{-\frac{\mathrm{i}t}{4k}} \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{Im} k < 0;$$

$$\tilde{J}(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{b_R(k)}{a_R(k)} & 1 \end{pmatrix} \begin{pmatrix} \frac{\bar{a}_R(\bar{k})}{\bar{A}(\bar{k})} & -b(k) \\ 0 & \frac{\bar{A}(\bar{k})}{\bar{a}_R(\bar{k})} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}} & 1 \end{pmatrix}, \qquad \operatorname{Im} k > 0.$$

Finally, for $|k| > |\mathcal{S}_\infty|$ we have

$$\tilde{J}(t,k) = \mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} \begin{pmatrix} \frac{a_R(k)}{A(k)} & 0 \\ -\bar{b}(\bar{k}) & \frac{A(k)}{a_R(k)} \end{pmatrix} \begin{pmatrix} \frac{1}{|a_R(k)|^2} & -\frac{b_R(k)}{a_R(k)} \\ -\frac{b_R(k)}{a_R(k)} & 1 \end{pmatrix} \begin{pmatrix} \frac{\bar{a}_R(\bar{k})}{\bar{A}(\bar{k})} & -b(k) \\ 0 & \frac{\bar{A}(\bar{k})}{\bar{a}_R(\bar{k})} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}}$$

$$= \begin{pmatrix} \frac{1}{|A(k)|^2} & -\frac{B(k)}{A(k)}\,\mathrm{e}^{-\frac{\mathrm{i}t}{4k}x} \\ -\frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\,\mathrm{e}^{\frac{\mathrm{i}t}{4k}} & 1 \end{pmatrix}, \qquad \operatorname{Im} k = 0.$$

The last equations show that $\tilde{J}(t,k) \equiv J^t(t,k)$ for $k \in \Gamma$. Therefore, the problem RH_{xt} for $x = 0$ and the problem RH_t are equivalent in the same sense as above. Further, the asymptotic relations

$$N(t,k) = m_0(t) + O(k),$$
$$M(0,t,k) = m(0,t) + O(k),$$
$$H(t,k) = I + O(k\,\mathrm{e}^{-t\frac{\operatorname{Im} k}{|k^2|}})$$

are fulfilled as $k \to 0$ $\quad(\operatorname{Im} k > 0)$. Taking into account the following formulae

$$\hat{Q}(t) = -m_0(t)\sigma_3 m_0^{-1}(t), \qquad m_0(t) = \lim_{k\to 0} M^{(t)}(t,k) = \lim_{k\to 0} N(t,k)$$
$$\hat{Q}(x,t) = -m(x,t)\sigma_3 m^{-1}(x,t), m(x,t) = \lim_{k\to 0} M(x,t,k)$$

we obtain

$$\hat{Q}(0,t) = -m(0,t)\sigma_3 m^{-1}(0,t) = -m_0(t)\sigma_3 m_0^{-1}(t) = \hat{Q}(t),$$

i.e., $\mu(0,t) = w(t)$ and $\nu(0,t) = v(t)$. □

Now we formulate the main result of the paper.

Theorem 5.1. *Let $u(x) \in H^1(0,L)$, $v(t), w(t) \in H^1(0,T)$ and $v^2(t) + |w(t)|^2 \equiv 1$. Then, the Riemann–Hilbert problem RH_{xt} has a unique solution $M(x,t,k)$, and the functions $q(x,t)$, $\nu(x,t)$ and $\mu(x,t)$ defined by the equations*

$$q(x,t) = 2\mathrm{i}\lim_{k\to\infty} kM_{12}(x,t,k),$$
$$\nu(x,t) = 1 - 2|m_{11}(x,t)|^2,$$
$$\mu(x,t) = -2\mathrm{i}m_{11}(x,t)m_{12}(x,t),$$

where

$$m(x,t) = \lim_{k\to 0} M(x,t,k),$$

satisfy the SRS equations (1), the initial condition

$$q(x,0) = u(x), \qquad x \in (0, L)$$

and boundary conditions

$$\nu(0,t) = v(t), \qquad \mu(0,t) = w(t), \qquad t \in (0, T).$$

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